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A change of variables in the asymptotic theory of differential equations with unbounded delays

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Abstract

In this paper, the author investigates the asymptotic properties of solutions of the nonhomogeneous linear differential equation

$$\dot{x}(t) = ax(\tau(t)) + bx(t) + f(t)$$

with nonzero real scalars a, b and the unbounded lag. Using the change of the independent and dependent variable he relates the asymptotic behaviour of solutions of this equation to the asymptotic behaviour of solutions of auxiliary functional (nondifferential) equations.

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1. Introduction

This paper is concerned with the functional differential equation (FDE)

$$\dot{x}(t) = ax(\tau(t)) + bx(t) + f(t), \tag{1.1}$$

where $a, b \neq 0$ are real constants, $\tau(t) \in C^1([t_0, \infty))$, $\tau(t)$ is mapping $[t_0, \infty)$ onto itself, $\tau(t) < t$ for all $t > t_0$, $0 < \dot{\tau}(t) \leqslant \lambda < 1$ for all $t \geqslant t_0$ and $f(t) \in C^0([t_0, \infty))$.

The simplest case of this type equation is the nonhomogeneous pantograph equation

$$\dot{x}(t) = ax(\lambda t) + bx(t) + f(t), \tag{1.2}$$

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which arises from the mathematical modelling of an industrial problem involving wave motion in the overhead supply line for a high speed train. The study of asymptotic properties of solutions of Eq. (1.2) was objective of Lim's paper [10]. Related works can also be found in Iserles [7], Liu [11], Pandolfi [14] and Shabat [16], where the authors investigated the more general FDEs with the proportional deviating argument.

Our aim is to generalize and extend results obtained in [10] to Eq. (1.1) with the more general type of the deviation. This type involves some important cases occurring in the applications as $\tau(t) = \lambda t + \rho$ or $\tau(t) = t^{\kappa}$. Among papers dealing with such FDEs we can mention, e.g., those of Derfel [4], Heard [6], Jaroš [8] and Čermák [2,3].

The main proof idea used in this paper is based on the change of the independent and dependent variable converting Eq. (1.1) into a more suitable form. The fundamental results of the transformation theory of FDEs can be found in works Neuman [12,13], Tryhuk [17], Zdun [18] and the above cited papers [6,8].

This paper is organized as follows. In Section 2, we introduce some necessary concepts and formulate main results. Section 3 is devoted to the proofs of the presented theorems. In the last section, we demonstrate some possible applications of our results and discuss a related topic.

2. Preliminaries and results

Throughout this paper we assume that real parameters a, b and real-valued functions $\tau(t)$, $f(t)$ fulfil the assumptions introduced in Section 1. By a solution of (1.1), we understand a real valued continuous function $x(t)$ defined in some subinterval $[\tau(\sigma), \infty)$ of $[t_0, \infty)$ and satisfying (1.1) for all $t \geqslant \sigma \geqslant t_0$. We note that without significant complication we can allow complex values for a and y while retaining b real.

If $\sigma = t_0$, then there exists a unique solution $x(t)$ of (1.1) defined for all $t \in [t_0, \infty)$ and satisfying the initial data $x(t_0) = x_0$. Let $\sigma > t_0$. If we are given a continuous function $\xi(t)$ defined in $[\tau(\sigma), \sigma]$, then there is a unique solution $x(t)$ of (1.1) defined for all $t \in [\tau(\sigma), \infty)$ and satisfying the condition $x(t) = \xi(t)$ for all $t \in [\tau(\sigma), \sigma]$.

The crucial role in our investigations of asymptotic properties of solutions of (1.1) plays Schröder equation

$$\varphi(\tau(t)) = \lambda\varphi(t), \tag{2.1}$$

where $\lambda > 0$, $\lambda \neq 1$ is a real parameter. In accordance with our previous notation we put

$$\lambda = \sup\{\dot{\tau}(t), t \geqslant t_0\}.$$

The theory of this equation is given in the monograph Kuczma et al. [9, Chapter 2]. Since we are interested in problems of the asymptotic behaviour, it is enough to consider solutions $\varphi(t)$ of (2.1) defined for $t \geqslant \sigma > t_0$. The existence theorems ensuring the existence of solutions defined on the whole $[t_0, \infty)$ require slightly modified assumptions (see [9]). However, as it was remarked above, it is not necessary to have such solutions in our investigations.

Before we state the relevant result we note that by the symbol $\tau^n(t)$ we mean the nth iterate of $\tau(t)$ (for positive integers n) or the nth iterate of the inverse function $\tau^{-1}(t)$ (for negative integers n).

Lemma 2.1. *Let $\sigma > t_0$ and let a function $\varphi_0(t) \in C^1([\tau(\sigma), \sigma])$ be such that*

$$\varphi_0(t) > 0, \quad 0 < \dot{\varphi}_0(t) \leqslant 1 \quad \text{for all } \tau(\sigma) \leqslant t \leqslant \sigma.$$

If $\varphi_0(\tau(\sigma)) = \lambda\varphi_0(\sigma)$ and $\dot{\varphi}_0(\tau(\sigma))\dot{\tau}(\sigma) = \lambda\dot{\varphi}_0(\sigma)$, then the formula

$$\varphi(t) = \lambda^{-n}\varphi_0(\tau^n(t)), \quad \tau^{-n+1}(\sigma) \leqslant t \leqslant \tau^{-n}(\sigma), \;\; n = 0, 1, 2, \ldots \tag{2.2}$$

defines a unique C^1 solution of (2.1) *on the interval $[\tau(\sigma), \infty)$ such that $\varphi(t) = \varphi_0(t)$ for any $t \in [\tau(\sigma), \sigma]$. Moreover, this solution fulfils the properties*

$$\varphi(t) > 0, \quad 0 < \dot{\varphi}(t) \leqslant 1 \quad \text{for all } t \geqslant \tau(\sigma). \tag{2.3}$$

Proof. The proof of the first part can be easily carried out using the step method. We prove inequalities (2.3).

Assume that $\dot{\varphi}(t^*) = 0$ for some $t^* > \sigma$ and let $\tau^{-m+1}(\sigma) \leqslant t^* \leqslant \tau^{-m}(\sigma)$ for a suitable integer m. Then $\dot{\varphi}(\tau^m(t^*)) = 0$, what contradicts the assumption $\dot{\varphi}_0(t) > 0$ for all $t \in [\tau(\sigma), \sigma]$.

Further,

$$\dot{\varphi}(t) = \frac{\dot{\tau}(t)}{\lambda}\dot{\varphi}(\tau(t)) \leqslant \dot{\varphi}(\tau(t)) \quad \text{for all } t \geqslant \sigma,$$

hence $\dot{\varphi}(t) \leqslant 1$ for all $t \geqslant \tau(\sigma)$. □

In the sequel, we put

$$\alpha = \frac{\log|a/(-b)|}{\log\lambda^{-1}}.$$

The main results can be summarized as follows.

Theorem 2.2. *Let $b > 0$, $f(t) = \mathrm{O}(\exp\{b\tau(t)\})$ as $t \to \infty$ and let $\varphi(t)$ be a solution of* (2.1) *given by* (2.2). *Then there exists a class $S = \{x_L(t),\ L \in \mathbb{R}\}$ of solutions of* (1.1) *possessing the property*

$$\exp\{-bt\}x_L(t) \to L \quad \text{as } t \to \infty. \tag{2.4}$$

Furthermore, for any solution $x(t)$ of (1.1) *there exists a particular solution $x_L(t) \in S$ and a function $g(t) = \mathrm{O}((\varphi(t))^\alpha)$ as $t \to \infty$ such that*

$$x(t) = x_L(t) + g(t), \quad t \text{ being large enough.} \tag{2.5}$$

Theorem 2.3. *Let $b < 0$, let $f(t) \in C^1([t_0, \infty))$ and let $\varphi(t)$ be a solution of* (2.1) *given by* (2.2). *If $f(t) = \mathrm{O}((\varphi(t))^\beta)$, $\dot{f}(t) = \mathrm{O}((\varphi(t))^{\beta-1})$ as $t \to \infty$ and $x(t)$ is any solution of* (1.1), *then*

$$x(t) = \mathrm{O}((\varphi(t))^\alpha) \quad \text{as } t \to \infty \quad \text{provided } \beta < \alpha. \tag{2.6}$$

$$x(t) = \mathrm{O}((\varphi(t))^\alpha \log\varphi(t)) \quad \text{as } t \to \infty \quad \text{provided } \beta = \alpha. \tag{2.7}$$

$$x(t) = \mathrm{O}((\varphi(t))^\beta) \quad \text{as } t \to \infty \quad \text{provided } \beta > \alpha. \tag{2.8}$$

3. Proofs

First, we state some auxiliary results that we need in the proofs of Theorems 2.2 and 2.3.

Lemma 3.1 (Pituk [15, Corollary 1]). *Consider the FDE*

$$\dot{z}(t) = p(t, z(\tau(t))), \tag{3.1}$$

where $\tau(t) \in C^0([t_0, \infty))$, $\tau(t) \leqslant t$ *for all* $t \geqslant t_0$, $\inf\{\tau(t), t \geqslant t_0\} > -\infty$. *Assume that* $p(t,z)$ *is a continuous function for which there exist a constant* C *and a continuous function* $r(t)$ *such that*

$$|p(t,z_1) - p(t,z_2)| \leqslant r(t)|z_1 - z_2| \tag{3.2}$$

and

$$|p(t,0)| \leqslant Cr(t) \tag{3.3}$$

for all $t \geqslant t_0$ *and all* $z_1, z_2 \in \mathbb{R}$. *If*

$$\int_{t_0}^{\infty} r(t)\,\mathrm{d}t < \infty, \tag{3.4}$$

then for every solution $z(t)$ *of* (3.1) *holds*

$$z(t) \to L \in \mathbb{R} \quad \text{as } t \to \infty.$$

Furthermore, if

$$\int_{t_0}^{\infty} r(t)\,\mathrm{d}t < 1, \tag{3.5}$$

then for every $L \in \mathbb{R}$ *there exists a solution* $z_L(t)$ *of* (3.1) *defined on* $[t_0, \infty)$ *such that*

$$z_L(t) \to L \quad \text{as } t \to \infty.$$

Lemma 3.2 (Carr and Dyson [1, Lemma 1] or Lim [10, Lemma 1]). *Let* $w(s)$ *be a solution of the difference equation*

$$w(s) - kw(s-c) = q(s), \tag{3.6}$$

where $c > 0$ *is a real constant*, k *is a complex constant such that* $|k| = \exp\{-\theta c\}$ *for a suitable* $\theta > 0$ *and* $q(s)$ *is a continuous function fulfilling*

$$q(s) = \mathrm{O}(\exp\{-\delta s\}) \quad \text{as } s \to \infty$$

for a suitable $\delta > 0$. *Then*

$$w(s) = \mathrm{O}(\exp\{-\theta s\}) \quad \text{as } s \to \infty \quad \text{provided } \theta < \delta, \tag{3.7}$$

$$w(s) = \mathrm{O}(s\exp\{-\theta s\}) \quad \text{as } s \to \infty \quad \text{provided } \theta = \delta, \tag{3.8}$$

$$w(s) = \mathrm{O}(\exp\{-\delta s\}) \quad \text{as } s \to \infty \quad \text{provided } \theta > \delta. \tag{3.9}$$

Lemma 3.3. *Let $\varphi(t)$ be a solution of* (2.1) *given by* (2.2) *and let $x(t)$ be a solution of the FDE*

$$\dot{x}(t) = m(t)x(\tau(t)) + bx(t) + n(t), \tag{3.10}$$

where $b < 0$ is a scalar, $m(t), n(t) \in C^0([t_0, \infty))$, $|m(t)| \leqslant M$ for all $t \geqslant t_0$ and $n(t) = \mathrm{O}((\varphi(t))^\mu)$ as $t \to \infty$. Then

$$x(t) = \mathrm{O}((\varphi(t))^\gamma) \quad \text{as } t \to \infty,$$

where $\gamma > \max(\mu, (\log M/(-b))/\log \lambda^{-1})$.

Proof. For the sake of simplicity, we can assume that a solution $x(t)$ of (3.10) is defined for all $t \geqslant \sigma > t_0$. We introduce the change of variables

$$s = \psi(t) = \log \varphi(t), \quad w(s) = (\varphi(t))^{-\gamma} x(t), \quad t \geqslant \sigma.$$

Then Eq. (3.10) becomes

$$w'(s) = m(h(s))\lambda^\gamma h'(s) w(s-c) + (bh'(s) - \gamma)w(s) + n(h(s)) \exp\{-\gamma s\} h'(s), \tag{3.11}$$

where $w'(s) = (\mathrm{d}w/\mathrm{d}s)(s)$, $h(s) \equiv \psi^{-1}(s)$ on $[\psi(\sigma), \infty)$ and $c = \log \lambda^{-1} > 0$. Eq. (3.11) can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}[\exp\{\gamma s - bh(s)\} w(s)] = {} & m(h(s))\lambda^\gamma h'(s) \exp\{\gamma s - bh(s)\} w(s-c) \\ & + \exp\{\gamma s - bh(s)\} n(h(s)) \exp\{-\gamma s\} h'(s).\end{aligned} \tag{3.12}$$

Choose $d_0 \geqslant \psi(\sigma)$ such that $\gamma - bh'(s) > 0$ for all $s \geqslant d_0$, put $d_j = d_0 + jc$, $I_j = [d_{j-1}, d_j]$, $j = 1, 2, \ldots$ and let $s \in I_{j+1}$. Integrating (3.12) over $[d_j, s]$ we obtain

$$\begin{aligned}\exp\{\gamma u - bh(u)\} w(u)|_{d_j}^s = {} & \int_{d_j}^s m(h(u))\lambda^\gamma h'(u) \exp\{\gamma u - bh(u)\} w(u-c)\,\mathrm{d}u \\ & + \int_{d_j}^s \exp\{\gamma u - bh(u)\} n(h(u)) \exp\{-\gamma u\} h'(u)\,\mathrm{d}u.\end{aligned}$$

Hence,

$$\begin{aligned}w(s) = {} & \exp\{\gamma(d_j - s) + b(h(s) - h(d_j))\} w(d_j) \\ & + \exp\{bh(s) - \gamma s\} \int_{d_j}^s m(h(u))\lambda^\gamma h'(u) \exp\{\gamma u - bh(u)\} w(u-c)\,\mathrm{d}u \\ & + \exp\{bh(s)) - \gamma s\} \int_{d_j}^s \exp\{\gamma u - bh(u)\} n(h(u)) \exp\{-\gamma u\} h'(u)\,\mathrm{d}u.\end{aligned}$$

Putting

$$B_j = \max\{|w(s)|, s \in I_j\}, \qquad j = 1, 2, \ldots,$$

we get

$$
\begin{aligned}
|w(s)| \leqslant\ & B_j \exp\{\gamma(d_j - s) + b(h(s) - h(d_j))\} \\
& + B_j \exp\{bh(s) - \gamma s\} \int_{d_j}^{s} |m(h(u))|\lambda^{\gamma} h'(u) \exp\{\gamma u - bh(u)\}\, \mathrm{d}u \\
& + \exp\{bh(s)) - \gamma s\} \int_{d_j}^{s} \exp\{\gamma u - bh(u)\} |n(h(u))| \exp\{-\gamma u\} h'(u)\, \mathrm{d}u.
\end{aligned} \tag{3.13}
$$

Substituting the relations

$$|m(h(u))|\lambda^{\gamma} \leqslant M\lambda^{\gamma} < -b$$

and

$$|n(h(u))| \exp\{-\gamma u\} \leqslant K_1 \exp\{(\mu - \gamma)u\} \quad \text{for a suitable } K_1 > 0,$$

into (3.13) we obtain

$$
\begin{aligned}
|w(s)| \leqslant\ & B_j \exp\{\gamma(d_j - s) + b(h(s) - h(d_j))\} \\
& + B_j |b| \exp\{bh(s) - \gamma s\} \int_{d_j}^{s} h'(u) \exp\{\gamma u - bh(u)\}\, \mathrm{d}u \\
& + K_1 \exp\{(\mu - \gamma)d_j\} \exp\{bh(s) - \gamma s\} \int_{d_j}^{s} h'(u) \exp\{\gamma u - bh(u)\}\, \mathrm{d}u.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|w(s)| \leqslant\ & B_j \exp\{\gamma(d_j - s) + b(h(s) - h(d_j))\} + \left(B_j + \frac{K_1}{|b|} \exp\{(\mu - \gamma)d_j\}\right) \\
& \times \exp\{bh(s) - \gamma s\} \int_{d_j}^{s} |b| h'(u) \exp\{\gamma u - bh(u)\}\, \mathrm{d}u.
\end{aligned} \tag{3.14}
$$

This integral can be written as

$$
\begin{aligned}
& \exp\{\gamma u - bh(u)\}|_{d_j}^{s} - \gamma \int_{d_j}^{s} \exp\{\gamma u - bh(u)\}\, \mathrm{d}u \\
& \quad \leqslant \exp\{\gamma u - bh(u)\}|_{d_j}^{s} + |\gamma| \int_{d_j}^{s} \exp\{\gamma u - bh(u)\}\, \mathrm{d}u.
\end{aligned}
$$

To estimate the last term we first note that

$$\frac{1}{h'(s)} = \frac{\dot{\varphi}(h(s))}{\varphi(h(s))} \leqslant \exp\{-s\}$$

by use of (2.3). Then the integration by parts yields

$$\begin{aligned}\int_{d_j}^{s} |\gamma| \exp\{\gamma u - bh(u)\}\,\mathrm{d}u &= \int_{d_j}^{s} \frac{|\gamma|}{\gamma - bh'(u)} \frac{\mathrm{d}}{\mathrm{d}u}[\exp\{\gamma u - bh(u)\}]\,\mathrm{d}u \\ &\leqslant K_2 \int_{d_j}^{s} \exp\{-u\} \frac{\mathrm{d}}{\mathrm{d}u}[\exp\{\gamma u - bh(u)\}]\,\mathrm{d}u \\ &\leqslant K_2 \exp\{-d_j\} \exp\{\gamma u - bh(u)\}|_{d_j}^{s}.\end{aligned}$$

Thus,

$$\int_{d_j}^{s} |b| h'(u) \exp\{\gamma u - bh(u)\}\,\mathrm{d}u \leqslant \exp\{\gamma u - bh(u)\}|_{d_j}^{s}(1 + K_2 \exp\{-d_j\}).$$

Substituting this into (3.14) we have

$$\begin{aligned}|w(s)| &\leqslant B_j \exp\{\gamma(d_j - s) + b(h(s) - h(d_j))\} \\ &\quad + \left(B_j + \frac{K_1}{|b|} \exp\{(\mu - \gamma)d_j\}\right) \exp\{bh(s) - \gamma s\} \\ &\quad \times \exp\{\gamma u - bh(u)\}|_{d_j}^{s}(1 + K_2 \exp\{-d_j\}) \\ &\leqslant B_j(1 + K_2 \exp\{-d_j\}) + \frac{K_1}{|b|} \exp\{(\mu - \gamma)d_j\}(1 + K_2 \exp\{-d_j\}) \\ &\leqslant B_j^*(1 + K_3 \exp\{-rd_j\}),\end{aligned}$$

where $B_j^* = \max(B_j, K_1/|b|)$, $r = \min(1, \gamma - \mu)$ and $K_3 > 0$ is a suitable constant.
This inequality holds for any $s \in I_{j+1}$, consequently

$$B_{j+1}^* \leqslant B_j^*(1 + K_3 \exp\{-rd_j\}) \leqslant B_1^* \prod_{i=1}^{j} (1 + K_3 \exp\{-rd_i\}), \quad j = 1, 2, \ldots .$$

Letting $j \to \infty$ we get the convergent infinite product

$$\prod_{i=1}^{\infty} (1 + K_3 \exp\{-rd_i\}).$$

Hence the sequences (B_j^*), resp. (B_j) are bounded as $j \to \infty$ and this proves the asymptotic property

$$x(t) = \mathrm{O}((\varphi(t))^{\gamma}) \quad \text{as } t \to \infty. \qquad \square$$

Proof of Theorem 2.2. First, we set $z(t) = \exp\{-bt\}x(t)$ in (1.1) to obtain

$$\dot{z}(t) = a \exp\{b(\tau(t) - t)\}z(\tau(t)) + f(t) \exp\{-bt\}, \tag{3.15}$$

what is the equation of form (3.1). Putting

$$r(t) \equiv |a| \exp\{b(\tau(t) - t)\} \quad \text{on } [t_0, \infty),$$

we can easily verify the validity of relations (3.2) and (3.4). Further,

$$|p(t,0)| = |f(t)| \exp\{-bt\} \leqslant C_1 \exp\{b\tau(t)\} \exp\{-bt\} = Cr(t),$$

what implies (3.3). Then by Lemma 3.1, every solution $z(t)$ of (3.15) tends to a finite (possibly zero) value as $t \to \infty$. Moreover, considering $\sigma_1 \geqslant t_0$ large enough we can fulfil (3.5) with t_0 replaced by σ_1. Thus for every finite constant L there exists a solution $z_L(t)$ of (3.15) defined on $[\sigma_1, \infty)$ and tending to L as $t \to \infty$. Now it is enough to put

$$S = \{x_L(t), x_L(t) \equiv \exp\{bt\} z_L(t) \text{ on } [\sigma_1, \infty), L \in \mathbb{R}\},$$

and the first part of Theorem 2.2 follows from Lemma 3.1.

To prove formula (2.5) we consider a solution $x(t)$ of (1.1) on the interval $[\sigma_2, \infty)$, $\sigma_2 > t_0$. Then by the previous part of the proof

$$\exp\{-bt\} x(t) \to L \quad \text{as } t \to \infty,$$

where L is a suitable real constant. Let us fix this L and consider the corresponding solution $x_L(t) \in S$ of (1.1) introduced above. We set

$$y(t) \equiv x(t) - x_L(t) \quad \text{on } [\sigma, \infty),$$

where $\sigma = \max(\sigma_1, \sigma_2)$. Then function $y(t)$ defines a solution of the homogeneous equation

$$\dot{y}(t) = a y(\tau(t)) + b y(t) \tag{3.16}$$

such that $y(t) = o(\exp\{bt\})$ as $t \to \infty$. Multiplying both sides of Eq. (3.16) by $\exp\{-bt\}$ we obtain

$$\frac{d}{dt}[\exp\{-bt\} y(t)] = a \exp\{-bt\} y(\tau(t)).$$

The integration over $[t, \infty)$ yields

$$y(t) = -a \exp\{bt\} \int_t^\infty \exp\{-bu\} y(\tau(u)) \, du.$$

Let $M_1 > 0$ be such that

$$|y(t)| \leqslant M_1 \exp\{bt\} \quad \text{for all } t \geqslant \sigma.$$

Then,

$$\begin{aligned} |y(t)| &\leqslant M_1 |a| \exp\{bt\} \int_t^\infty \exp\{b(\tau(u) - u)\} \, du \\ &\leqslant M_1 \frac{|a|}{b(1-\lambda)} \exp\{bt\} \exp\{b(\tau(t) - t)\} = M_1 \frac{|a|}{b(1-\lambda)} \exp\{b\tau(t)\} \\ &\leqslant M_1 \frac{|a|}{b(1-\lambda)} \exp\{b\tau^{-1}(\sigma)\} \quad \text{for all } \tau^{-1}(\sigma) \leqslant t \leqslant \tau^{-2}(\sigma). \end{aligned}$$

Repetition leads to

$$|y(t)| \leqslant M_1 \frac{|a|^n}{b^n(1-\lambda)\dots(1-\lambda^n)} \exp\{b\tau^n(t)\}$$

$$\leqslant M_1 \frac{|a|^n}{b^n(1-\lambda)\dots(1-\lambda^n)} \exp\{b\tau^{-1}(\sigma)\}$$

for all $\tau^{-n}(\sigma) \leqslant t \leqslant \tau^{-n-1}(\sigma)$.

Now, let $M_2 > 0$ be defined as

$$\inf\{(\varphi(t))^\alpha, \quad \sigma \leqslant t \leqslant \tau^{-1}(\sigma)\}.$$

Then

$$(\varphi(t))^\alpha \geqslant M_2 \frac{|a|}{b} \quad \text{for all } \tau^{-1}(\sigma) \leqslant t \leqslant \tau^{-2}(\sigma),$$

and, generally,

$$(\varphi(t))^\alpha \geqslant M_2 \frac{|a|^n}{b^n} \quad \text{for all } \tau^{-n}(\sigma) \leqslant t \leqslant \tau^{-n-1}(\sigma).$$

Thus,

$$|y(t)| \leqslant M_3(\varphi(t))^\alpha \quad \text{for all } t \geqslant \sigma,$$

where $M_3 = M_1 \exp\{b\tau^{-1}(\sigma^*)\}/(M_2 \prod_{i=1}^{\infty}(1-\lambda^i)) > 0$.

This proves formula (2.5). □

Proof of Theorem 2.3. Let $\gamma > \max(\beta, \alpha)$. We set

$$s = \psi(t) = \log \varphi(t), \quad w(s) = (\varphi(t))^{-\gamma} x(t), \quad t \geqslant \sigma > t_0$$

in (1.1) to obtain the equation

$$w'(s) = a\lambda^\gamma h'(s) w(s-c) + (bh'(s) - \gamma) w(s) + f(h(s)) \exp\{-\gamma s\} h'(s), \tag{3.17}$$

where $h(s)$ and c have the same meaning as in the proof of Lemma 3.3. By this assertion, $x(t) = \mathrm{O}((\varphi(t))^\gamma)$ as $t \to \infty$, i.e., $w(s)$ is bounded. Differentiating (1.1) we obtain

$$\ddot{x}(t) = a\dot{\tau}(t)\dot{x}(\tau(t)) + b\dot{x}(t) + \dot{f}(t).$$

This equation has form (3.10) with $M = |a|\lambda$ and $\mu = \beta - 1$. The repeated application of Lemma 3.3 yields $\dot{x}(t) = \mathrm{O}((\varphi(t)^{\gamma-1})$ as $t \to \infty$. Consequently,

$$|w'(s)| = |\gamma| \exp\{-\gamma s\}|x(t)| + \exp\{-\gamma s\}|\dot{x}(t)| h'(s) \leqslant N_1 \exp\{-s\} h'(s).$$

Rewrite Eq. (3.17) as (3.6), where

$$k = \frac{a\lambda^\gamma}{-b}, \quad q(s) = \frac{1}{bh'(s)}[w'(s) + \gamma w(s) - f(h(s)) \exp\{-\gamma s\} h'(s)],$$

i.e., $\theta = \gamma - \alpha > 0$ and $|q(s)| \leqslant N_2 \exp\{-s\} + N_3 \exp\{(\beta - \gamma)s\}$. Now we distinguish three cases with the respect to the relation between β and α.

Case (2.6): First let $\beta < \alpha < \beta + 1$. Then

$$q(s) = \mathrm{O}(\exp\{(\beta - \gamma)s\}) \quad \text{as } s \to \infty,$$

for a suitable γ, $\alpha < \gamma < \beta + 1$. Hence by (3.7) (with $\delta = \gamma - \beta > \theta$) $w(s) = \mathrm{O}(\exp\{(\alpha - \gamma)s\})$ as $s \to \infty$, what implies (2.6).

If $\alpha \geqslant \beta + 1$, then we can choose any γ, $\alpha < \gamma < \alpha + 1$ and

$$q(s) = \mathrm{O}(\exp\{-s\}) \quad \text{as } s \to \infty,$$

i.e., $\delta = 1$. Asymptotic relation (2.6) now follows from (3.7).

Case (2.7): Let $\beta = \alpha$. Choose any γ, $\alpha < \gamma < \alpha + 1$ and notice that

$$q(s) = \mathrm{O}(\exp\{(\beta - \gamma)s\}) \quad \text{as } s \to \infty.$$

Thus $w(s) = \mathrm{O}(s \exp\{(\alpha - \gamma)s\})$ as $s \to \infty$ by use of (3.8), what implies (2.7).

Case (2.8): Let $\beta > \alpha$. Then

$$q(s) = \mathrm{O}(\exp\{(\beta - \gamma)s\}) \quad \text{as } s \to \infty$$

for a suitable γ, $\beta < \gamma < \beta + 1$. Hence $\delta = \gamma - \beta$ and (2.8) follows from (3.9). This completes the proof of Theorem 2.3. $\square$

4. Applications

In this section, we give some examples illustrating the conclusions of Theorems 2.2 and 2.3. To apply the relevant asymptotic formulas we need to solve Schröder equation (2.1). Generally speaking, this equation can be solved by the step method (see Lemma 2.1). Nevertheless, in several important cases we are able to give the explicit form of the solution $\varphi(t)$ of (2.1) having all the properties stated in Lemma 2.1.

Example 4.1. We consider the delay equation with the linearly transformed argument

$$\dot{x}(t) = ax(\lambda t + \rho) + bx(t) + f(t), \quad 0 < \lambda < 1, \quad \rho \in \mathbb{R}, \tag{4.1}$$

where $t \in [\rho/(1-\lambda), \infty)$. Equations with this type of the delay serve as a mathematical model of the various problems ranging from the number theory to astrophysics (see, e.g., [4]).

Schröder equation (4.1) becomes

$$\varphi(\lambda t + \rho) = \lambda \varphi(t)$$

and admits the solution $\varphi(t) = t - \rho/(1-\lambda)$. Clearly, $\varphi(t) > 0$ for any $t > \rho/(1-\lambda)$ and $\dot{\varphi}(t) \equiv 1$ on $[\rho/(1-\lambda), \infty)$. Thus $\varphi(t)$ has all the properties mentioned in Lemma 2.1.

Now let $b > 0$, $f(t) \in C^0([\rho/(1-\lambda), \infty))$, $f(t) = \mathrm{O}(\exp\{b\lambda t\})$ as $t \to \infty$. Then for any $L \in \mathbb{R}$ there exists a particular solution $x_L(t)$ of (4.1) fulfilling the property (2.4). Moreover, any solution $x(t)$ of (4.1) can be represented in form (2.5), where the real constant L and the function $g(t) = \mathrm{O}(t^{\alpha})$ as $t \to \infty$ are depending on $x(t)$.

Let $b < 0$, $f(t) \in C^1([\rho/(1-\lambda), \infty))$, $f(t) = \mathrm{O}(t^\beta)$ and $\dot{f}(t) = \mathrm{O}(t^{\beta-1})$ as $t \to \infty$. If $x(t)$ is a solution of (4.1), then:

$$x(t) = \mathrm{O}(t^\alpha) \quad \text{as } t \to \infty \quad \text{provided } \beta < \alpha.$$

$$x(t) = \mathrm{O}(t^\alpha \log t) \quad \text{as } t \to \infty \quad \text{provided } \beta = \alpha.$$

$$x(t) = \mathrm{O}(t^\beta) \quad \text{as } t \to \infty \quad \text{provided } \beta > \alpha.$$

Notice that putting $\rho = 0$ in (4.1) we obtain the pantograph equation (1.2). Then our previous results generalize (the case $b < 0$) or improve (the case $b > 0$) those of Lim [10].

Example 4.2. Quite similarly we can investigate the asymptotic behaviour of solutions of the delay equation

$$\dot{x}(t) = ax(t^\kappa) + bx(t) + f(t), \quad 0 < \kappa < 1, \tag{4.2}$$

where $t \in [1, \infty)$. Now $\lambda = \kappa$ and the corresponding Schröder equation (2.1) can be read as

$$\varphi(t^\kappa) = \kappa\varphi(t).$$

This equation admits the function $\varphi(t) = \log t$ as the solution with the required properties. Substituting this $\varphi(t)$ into Theorems 2.2 and 2.3 we obtain the corresponding asymptotic formulas for all solutions of Eq. (4.2).

Example 4.3. We consider the delay equation

$$\dot{x}(t) = b[x(t) - x(\tau(t))] + f(t), \tag{4.3}$$

where $\tau(t)$ and $f(t)$ satisfy the assumptions introduced in Section 1.

First let $b > 0$ and $f(t) = \mathrm{O}(\exp\{b\tau(t)\})$ as $t \to \infty$. Then there exists a class of particular solutions $x_L(t)$, $L \in \mathbb{R}$ of (4.3) possessing the property (2.4). Further notice that due to $\alpha = 0$ it is not necessary to solve Eq. (2.1) when applying conclusions of Theorem 2.2. Consequently, for any solution $x(t)$ of (4.3) there exists a solution $x_L(t)$ of (4.3) and a bounded function $g(t)$ such that formula (2.5) holds.

Now let $b < 0$. Applying Theorem 2.3 we have to distinguish three cases with the respect to the behaviour of $f(t) \in C^1([t_0, \infty))$. Assume, e.g., that $f(t)$ is bounded (i.e., $\beta = \alpha = 0$). Then by (2.7) each solution $x(t)$ of (4.3) satisfies the relation

$$x(t) = \mathrm{O}(\log \varphi(t)) \quad \text{as } t \to \infty,$$

where $\varphi(t)$ is a solution of (2.1) given by (2.2). The remaining two cases can be dealt with quite similarly.

We note that Eq. (4.3) and its modifications have been studied by many authors (for results and references see Diblík [5]). Our conclusions generalize some of these results.

Our final remark discusses an interesting relationship between solutions of Eq. (1.1) and a solution of an auxiliary functional (nondifferential) equation. We demonstrate this connection under the assumption $b < 0$.

We introduce the function $\omega_1(t) \equiv (\varphi(t))^\alpha$ occurring in the O-estimate (2.6). It is easy to check that $\omega_1(t)$ defines a solution of the functional equation

$$|a|\,\omega_1(\tau(t)) + b\omega_1(t) = 0. \tag{4.4$_1$}$$

Similarly, we put $\omega_2(t) \equiv (\varphi(t))^\alpha \log \varphi(t)$ and verify that

$$|a|\omega_2(\tau(t)) + b\omega_2(t) + f^*(t) = 0, \quad f^*(t) = \mathrm{O}((\varphi(t))^\alpha) \quad \text{as } t \to \infty \tag{4.4$_2$}$$

(more precisely $f^*(t) = b \log \lambda (\varphi(t))^\alpha$).

Finally, let $\omega_3(t) \equiv (\varphi(t))^\beta$. Then $\omega_3(t)$ satisfies the functional equation

$$|a|\,\omega_3(\tau(t)) + b\omega_3(t) + f^{**}(t) = 0, \quad f^{**}(t) = \mathrm{O}((\varphi(t))^\beta) \quad \text{as } t \to \infty \tag{4.4$_3$}$$

(more precisely $f^{**}(t) = b(\lambda^{\beta-\alpha} - 1)(\varphi(t))^\beta$).

Now we can reformulate the O-estimates (2.6)–(2.8) of solutions $x(t)$ of (1.1) derived in Theorem 2.3 as follows:

$$x(t) = \mathrm{O}(\omega_1(t)) \text{ as } t \to \infty \quad \text{provided } f(t) = \mathrm{O}(\varphi(t))^\beta, \quad \beta < \alpha.$$

$$x(t) = \mathrm{O}(\omega_2(t)) \text{ as } t \to \infty \quad \text{provided } f(t) = \mathrm{O}(\varphi(t))^\beta, \quad \beta = \alpha.$$

$$x(t) = \mathrm{O}(\omega_3(t)) \text{ as } t \to \infty \quad \text{provided } f(t) = \mathrm{O}(\varphi(t))^\beta, \quad \beta > \alpha.$$

Consequently, each solution $x(t)$ of the delay differential equation (1.1) can be estimated by means of a solution $\omega_i(t)$ of a functional equation (4.4$_i$), $i = 1, 2, 3$.

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References

[1] J. Carr, J. Dyson, The matrix functional differential equation $y'(x) = Ay(\lambda x) + By(x)$, Proc. Roy. Soc. Edinburgh 75A (1975/76) 5–22.
[2] J. Čermák, The asymptotic bounds of solutions of linear delay systems, J. Math. Anal. Appl. 225 (1998) 373–388.
[3] J. Čermák, On the delay differential equation $y'(x) = ay(\tau(x)) + by(x)$, Ann. Polon. Math. LXXI.2 (1999) 161–169.
[4] G.A. Derfel, Functional-differential equations with compressed arguments and polynomial coefficients: asymptotic of solutions, J. Math. Anal. Appl. 193 (1995) 671–679.
[5] J. Diblík, Asymptotic representation of solutions of equation $\dot{y}(t) = \beta(t)[y(t) - y(t - \tau(t))]$, J. Math. Anal. Appl. 217 (1998) 200–215.
[6] M.L. Heard, A change of variables for functional-differential equations, J. Differential Equations 18 (1975) 1–10.
[7] A. Iserles, On the generalized pantograph functional-differential equation, European J. Appl. Math. 4 (1992) 1–38.
[8] J. Jaroš, An application of change of independent variable in the oscillation theory of differential equations with unbounded delays, Acta Math. Univ. Com. LVIII–LIX (1990) 99–106.
[9] M. Kuczma, B. Choczewski, R. Ger, Iterative Functional Equations, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1990.
[10] E.B. Lim, Asymptotic bounds of solutions of the functional differential equation $x'(t) = ax(\lambda t) + bx(t) + f(t), 0 < \lambda < 1$, SIAM J. Math. Anal. 9 (1978) 915–920.

[11] Y. Liu, Regular solutions of the Shabat equation, J. Differential Equations 154 (1999) 1–41.
[12] F. Neuman, On transformations of differential equations and systems with deviating argument, Czechoslovak Math. J. 31 (106) (1981) 87–90.
[13] F. Neuman, Transformations and canonical forms of functional-differential equations, Proc. Roy. Soc. Edinburgh 115A (1990) 349–357.
[14] L. Pandolfi, Some observations on the asymptotic behaviors of the solutions of the equation $x'(t) = A(t)x(\lambda t) + B(t)x(t), \lambda > 0$, J. Math. Anal. Appl. 67 (1979) 483–489.
[15] M. Pituk, On the limits of solutions of functional differential equations, Math. Bohemica 118 (1993) 53–66.
[16] A. Shabat, The infinite dimensional dressing dynamical system, Inverse Problems 8 (1992) 303–308.
[17] V. Tryhuk, Pointwise transformations of linear functional-differential equations of the nth order, Fasciculi Math. 27 (1997) 95–100.
[18] M. Zdun, On simultaneous Abel equations, Aequationes Math. 38 (1989) 163–177.